A generalized family of discrete $\mathcal{PT}-$symmetric square wells

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Abstract

$N-$site-lattice Hamiltonians $H^{(N)}$ are introduced and perceived as a set of systematic discrete approximants of a certain $\mathcal{PT}-$symmetric square-well-potential model with the real spectrum and with a non-Hermiticity which is localized near the boundaries of the interval. Its strength is controlled by one, two or three parameters. The problem of the explicit construction of a nontrivial metric which makes the theory unitary is then addressed. It is proposed and demonstrated that due to the not too complicated (viz., tridiagonal matrix) form of our input Hamiltonians, the computation of the metric is straightforward and that its matrix elements prove obtainable, non-numerically, in elementary polynomial forms.

Keywords

quantum mechanics; discrete lattices; non-Hermitian Hamiltonians; Hilbert-space metrics; solvable models;

1 Introduction

A priori it is clear that the traditional and most common physical Hilbert spaces of the admissible quantum states need not necessarily prove optimal for computations. Once these "obvious" spaces $\mathcal{H}^{(P)}$ become distinguished by the superscript $^{(P)}$ which may be read as an abbreviation for "primary space", one may find an explicit verification of this expectation in nuclear physics cca twenty years ago [1]. The amended Schrödinger-representation Hilbert space $\mathcal{H}^{(S)}$ (where the superscript stands for "secondary") has been constructed there via a fermion-boson-space correspondence $P \leftrightarrow S$.

A perceivable simplification of the practical numerical evaluation and/or at least of the variational prediction of the bound-state energy levels E_n has been achieved for a number of heavy nuclei. In the notation as introduced in Ref. [2] one can identify the underlying key mathematical idea as lying in a Dyson-inspired ansatz connecting the $P-$superscripted and $S-$superscripted ket-vectors,

$$|\psi^{(P)}\rangle \ = \ \Omega\,|\psi^{(S)}\rangle \in \mathcal{H}^{(P)}\,, \qquad |\psi^{(S)}\rangle \in \mathcal{H}^{(S)}\,. \tag{1}$$

The manipulations with the original ket vectors $|\psi^{(P)}\rangle$ became, by such a construction, facilitated.

In particular, what appeared simplified was the evaluation of the inner products $\langle\phi^{(P)}|\psi^{(P)}\rangle$ and of the $P-$space matrix elements, say, of the Hamiltonian operator $\mathfrak{h}$ acting in $\mathcal{H}^{(P)}$. After the unitary-equivalence transition to $\mathcal{H}^{(S)}$ the same quantities were represented by the new inner products $\langle\phi^{(S)}|\psi^{(S)}\rangle$ and by the matrix elements $\langle\phi^{(S)}|H|\psi^{(S)}\rangle$, respectively.

It is well known [3, 4, 5] that during the transition $P \leftrightarrow S$ between Hilbert spaces one must also guarantee the isospectrality between the respective Hamiltonians $\mathfrak{h}$ and H. In other words,, we must define the new

Hamiltonian H acting in $\mathcal{H}^{(S)}$ by formula $H = \Omega^{-1}\mathfrak{h}\Omega$. Then, it appears natural when the whole change of the representation $P \to S$ is followed by another, second-step simplification. Such a step is usually motivated by the survival of certain cumbersome character of the work in the secondary Hilbert space $\mathcal{H}^{(S)}$. In the notation of Ref. [2], for example, it makes sense to replace the latter space by its "friendlier", auxiliary, manifestly unphysical alternative $\mathcal{H}^{(F)}$.

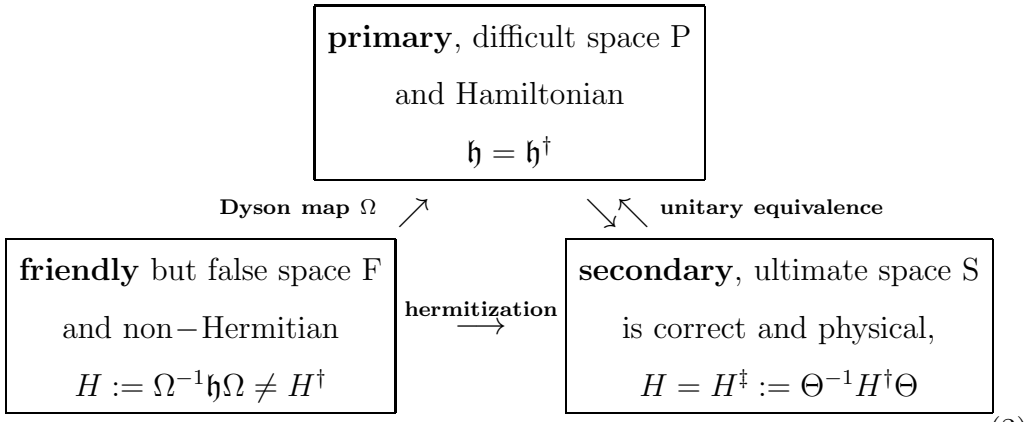
Due to a certain freedom in the construction, the latter, third Hilbert space may be allowed to coincide with $\mathcal{H}^{(S)}$ as a topological vector space (i.e., as the space of kets, $|\psi^{(F)}\rangle| := \psi^{(S)}\rangle$). What leads to the ultimate simplicity is then the replacement of the fairly complicated, $S-$superscripted operation $\mathcal{T}^{(S)}$ of the Hermitian conjugation in $\mathcal{H}^{(S)}$ by the standard and trivial (i.e., transposition plus complex conjugation) $F-$superscripted operation $\mathcal{T}^{(F)}$ of the Hermitian conjugation in the final friendly space $\mathcal{H}^{(F)}$.

The net purpose of the second simplification step $S \to F$ is that the quantum system in question finds its optimal Schrödinger representation in $\mathcal{H}^{(F)}$. In this auxiliary and maximally friendly Hilbert space one merely defines

$$\langle \phi^{(S)}|\psi^{(S)}\rangle \ \equiv \ \langle \phi^{(F)}|\Theta|\psi^{(F)}\rangle\,, \qquad \Theta = \Omega^\dagger\Omega \tag{2}$$

This convention keeps trace of the $S-$superscripted definition of the physics-representing inner products in $\mathcal{H}^{(S)}$ and it offers a guarantee of validity of the initial requirement of the unitary equivalence between $\mathcal{H}^{(P)}$ and $\mathcal{H}^{(S)}$. In a compact review [2] of the formalism we emphasized that a given quantum bound-state system is in fact characterized by a *triplet* of Hilbert spaces

according to the following diagram:



(3)

During the above-mentioned application of such a pattern to the variational analysis of heavy nuclei it has been emphasized that, firstly, the model itself is introduced in the P-superscripted Hilbert space but it appeared there prohibitively complicated [1]. Secondly, the successful choices of the suitable simplification mappings Ω have been found dictated or inspired by the underlying dynamics (i.e., in nuclei, by the tendency of fermions to form, effectively, certain boson-resembling clusters). Thirdly, in a way reaching far beyond the particular nuclear physics context, the product $\Omega^\dagger\Omega = \Theta \neq I$ has been noticed to play the role of the metric in the ultimate, S-superscripted Hilbert-space.

Cca ten years ago, the metric-operator interpretation of nontrivial $\Theta \neq I$ became believed to apply to a very broad family of models including, typically, the imaginary-cubic oscillator

$$H = -\frac{d^2}{dx^2} + \mathrm{i}x^3 \tag{4}$$

as well as many other Hamiltonians H introduced as acting in $\mathcal{H}^{(F)} := L^2(\mathbb{R})$ and/or in $\mathcal{H}^{(S)} \neq L^2(\mathbb{R})$ and relevant, typically, in the relativistic quantum field theory (cf., e.g., [4] or [5] for extensive details).

The basic ideas behind the pattern of Eq. (3) were broadly accepted and

the whole mathematical formalism (which we call, conveniently, the three-Hilbert-space (THS) representation of quantum states) started to be treated as an old and well understood one. In the year 2012, this opinion has rather drastically been challenged by the results of Refs. [6] where it has been proved, *rigorously*, that for the most popular "benchmark" THS model (4) the class of the eligible Hilbert-space metric operators Θ is in fact *empty*. In other words we were all suddenly exposed to the necessity of reanalyzing the mathematics behind the differential-operator models as sampled by Eq. (4).

This observation belongs to one of the key motivations of our present study. The emergence of incompatibility of the overall methodical THS pattern (3) with the concrete unbounded-operator example (4) implies that the attention of mathematical physicists must immediately be redirected and returned to the alternative, mathematically correct benchmark models like, e.g., the bounded-operator Hamiltonians of Ref. [1] and/or even to the most schematic, exactly solvable finite-dimensional models as sampled, say, by the non-numerical discrete square well of our preceding Paper 1 [7].

The latter family of models was characterized by the sequence of the most elementary finite-dimensional Hamiltonians

$$H^{(3)}(\lambda) = \begin{bmatrix} 2 & -1-\lambda & 0 \\ -1+\lambda & 2 & -1+\lambda \\ 0 & -1-\lambda & 2 \end{bmatrix},$$

$$H^{(4)}(\lambda) = \begin{bmatrix} 2 & -1-\lambda & 0 & 0 \\ -1+\lambda & 2 & -1 & 0 \\ 0 & -1 & 2 & -1+\lambda \\ 0 & 0 & -1-\lambda & 2 \end{bmatrix}$$

$$H^{(5)}(\lambda)=\left[\begin{array}{ccccc}
2 & -1-\lambda & 0 & 0 & 0 \\
-1+\lambda & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1+\lambda \\
0 & 0 & 0 & -1-\lambda & 2
\end{array}\right]$$

i.e., by the matrix

$$H^{(N)}(\lambda)=\left[\begin{array}{cccccc}
2 & -1-\lambda & 0 & \ldots & 0 & 0 \\
-1+\lambda & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & \ddots & \ddots & \vdots \\
\vdots & 0 & \ddots & \ddots & -1 & 0 \\
0 & \vdots & \ddots & -1 & 2 & -1+\lambda \\
0 & 0 & \ldots & 0 & -1-\lambda & 2
\end{array}\right] \tag{5}$$

considered at an arbitrary preselected Hilbert-space dimension N. As required, this matrix appears non-Hermitian in the $N-$dimensional and manifestly unphysical, auxiliary (and, in our case, real) Hilbert space $\mathcal{H}^{(F)}_{(N)} \equiv \mathbb{R}^N$ where the inner product remains trivial,

$$\langle \phi^{(F)} | \psi^{(F)} \rangle = \sum_{n=1}^{N} \phi_n^{(F)} \, \psi_n^{(F)} \, .$$

In Paper 1 we emphasized that one may try to deduce the physical context, contents and meaning of models (5) in their $N \to \infty$ limiting coincidence with certain usual single-parametric differential Schrödinger operators on the line [8].

In the additional, methodical role of non-contradictory and exactly solvable, non-numerical benchmark models, the most serious weakness of Hamiltonians (5) may be seen in their trivial kinetic-operator nature inside the whole interior of the interval of the spatial coordinate x (see also Paper 1 for a more explicit explanation and further references). This means that their

nontrivial dynamical content (i.e., their point-like-interaction component) is merely one-parametric and restricted to the points of the spatial boundary.

In our present paper we intend to extend this perspective in a systematic manner by showing, first of all, that the latter weakness of the models of Paper 1 is curable. We shall introduce and employ a few less elementary toy-model interactions on the same $N-$site quantum lattice. In Section 2 we select just a less trivial version of the one-parametric interaction while in subsequent Sections 3 and 4, two and three parameters controlling the interaction are introduced, respectively. Our overall message is finally summarized in Sections 5 (discussion and outlook) and 6 (summary).

2 A slightly more sophisticated one-parametric model

2.1 Hamiltonians $H^{(N)}$ and metrics $\Theta^{(N)}$

Let us consider a non-Hermitian and real N by N Hamiltonian matrix H in which the interaction connects the triplets of the next-to-the-boundary sites,

$$H^{(N)}(\lambda)=\left[\begin{array}{ccccccccc}
2 & -1-\lambda & 0 & 0 & \ldots & & \ldots & 0 \\
-1+\lambda & 2 & -1+\lambda & 0 & \ldots & & & \vdots \\
0 & -1-\lambda & 2 & -1 & \ddots & & & \\
0 & 0 & -1 & 2 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & -1 & 0 & 0 \\
 & & & \ddots & -1 & 2 & -1-\lambda & 0 \\
\vdots & & & \ldots & 0 & -1+\lambda & 2 & -1+\lambda \\
0 & \ldots & & \ldots & 0 & 0 & -1-\lambda & 2
\end{array}\right]. \tag{6}$$

Recalling the experience gained in Paper 1 we may expect that the bound-state-energy eigenvalues obtained from this Hamiltonian will be all real at the sufficiently small values of the couplings $\lambda \in (-a, a)$ with, presumably, $a = 1$.

A rigorous proof of the above conjecture would be feasible albeit lengthy. Although we are not going to present it here due to the lack of space, Figure 1 samples the whole spectrum at $N = 11$ and offers a persuasive numerical support of such an expectation. Moreover, a comparison of this picture with its predecessors of Paper 1 indicates that the use of a less trivial Hamiltonian seems truly rewarding. In the past, the phenomenologically rich and promising nontrivial structure of the parameter-dependence of the spectrum near $\lambda \approx a$ motivated quite strongly the continuation of our study of similar, more complicated toy models.

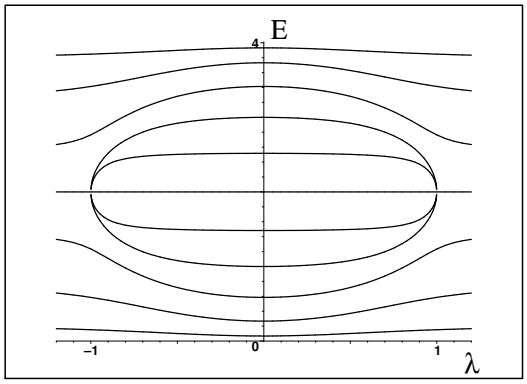


Figure 1: The $\lambda-$dependence of the eigenvalues of Hamiltonian (6). Obviously, this spectrum stays real in the interval of $\lambda \in (-1, 1)$.

Under the hypothesis of the reality of the spectrum, a completion of the construction of the corresponding consistent THS quantum model requires, naturally, the explicit construction of a metric Θ entering the physical inner product (2). In its full completeness, such a task has been pursued in Paper 1. In what follows we intend to complement this research towards some

more complicated Hamiltonians sampled by Eq. (6) above. At the same time we shall skip all details of an exhaustive analysis and reduce the exhaustive constructive classification of the $N-$parametric sets of metrics

$$\Theta^{(N)} = \sum_{k=1}^{N} \mu_k \, \mathcal{P}^{(k)} \tag{7}$$

to the mere evaluation of a characteristic sample of its individual Hermitian-matrix components $\mathcal{P}_k^{(N)}$. These components may be interpreted as metric-resembling (i.e., not necessarily positive definite) matrices. Their main pedagogical merit is that they remain sufficiently transparent matrices with, hopefully, sparse structure of the universal form which has been found and described in Paper 1.

With this purpose in mind we shall require that the individual components of the sum Eq. (7) satisfy the Dieudonné equation *alias* quasi-Hermiticity condition

$$\sum_{m=1}^{N} \left[\left(H^{\dagger}\right)_{jm} \mathcal{P}_{mn} - \mathcal{P}_{jm} \, H_{mn} \right] = 0 \,, \qquad j,n = 1,2,\ldots,N \,. \tag{8}$$

In the light of the analysis of Paper 1 we shall, furthermore, save time and skip the exhaustive discussion of the (more or less trivial) general $N-$dependence of the model. In order to gain an overall insight into the structure of the THS representability of our model, we found it sufficient to restrict attention to a fixed value of dimension N which is neither too small (we have to avoid the structural degeneracies at small N) nor too large (we intend to display some matrices in print).

2.2 Matrix $\mathcal{P}^{(6)}$ at $N = 11$

Following the recipe described in Paper 1 we shall start from the ansatz

$$\mathcal{P}^{(6)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s & 0 & s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v & 0 & t & 0 & v & 0 & 0 & 0 \\
0 & 0 & v & 0 & w & 0 & w & 0 & v & 0 & 0 \\
0 & s & 0 & w & 0 & 1 & 0 & w & 0 & s & 0 \\
r & 0 & t & 0 & 1 & 0 & 1 & 0 & t & 0 & r \\
0 & s & 0 & w & 0 & 1 & 0 & w & 0 & s & 0 \\
0 & 0 & v & 0 & w & 0 & w & 0 & v & 0 & 0 \\
0 & 0 & 0 & v & 0 & t & 0 & v & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s & 0 & s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{9}$$

and, in the light of Eq. (8), we shall compare the matrix product $\mathcal{P}^{(6)}\,H$ with the matrix product $H^{\dagger}\mathcal{P}^{(6)}$. Element by element, their (row-wise running) comparison yields the nontrivial constraints $s = s\lambda + r$ in the fifth and seventh step, $v = -v\lambda + s$ in the fifteenth step, etc. After the tedious though entirely straightforward manipulations we obtain the final solution/formulae

$$r = \frac{1-\lambda^2}{1+3\,\lambda^2}\,, \quad s = \frac{1+\lambda}{1+3\,\lambda^2}\,, \quad v = \frac{1}{1+3\,\lambda^2}$$

$$t = \frac{1+\lambda^2}{1+3\,\lambda^2}\,, \quad w = \frac{1+2\,\lambda^2}{1+3\,\lambda^2} \tag{10}$$

which indicate that the transition to the more-site interactions in the Hamiltonian may still be expected to lead to the polynomial or rational-function dependence of the matrix elements of the metric on the value of the coupling constant. The second, methodically equally encouraging consequence of the construction of the sample pseudometric $\mathcal{P}^{(6)}$ is that after a not too drastic

loss of the simplicity of the input matrix Hamiltonians the construction of the class of admissible metric remains feasible by non-numerical means. Thirdly, via a deeper analysis of Dieudonné's Eq. (8) it is easy to deduce that the extension of the $N = 11$ results to any dimension $N > 11$ parallels the pattern found in Paper 1 and degenerates to a virtually trivial extrapolation of the interior parts of individual items $\mathcal{P}^{(k)}$ in the matrix sequences determining the general metric (7).

3 Two-parametric Hamiltonians

3.1 Energies

Once we recall preceding section and disentangle the values of the respective couplings between the two next-to-boundary and two next-to-next-to-boundary sites we obtain the following two-parametric $N = 11$ Hamiltonian matrix

$$\left[\begin{array}{cccccccccccc}
2 & -1-\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1+\lambda & 2 & -1+\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1-\mu & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1-\mu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\mu & 2 & -1+\lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1-\lambda & 2
\end{array}\right] \tag{11}$$

Its full display still almost fits in the printed page but what is certainly more important is that the presence of the new variable coupling μ extends the capability of the model of being more useful in some phenomenologically oriented considerations. This seems well illustrated by Fig. 2 where we restricted attention to a line in the plane of parameters defined by the constraint $\mu \to \mu(\lambda) = \lambda +$ a constant.

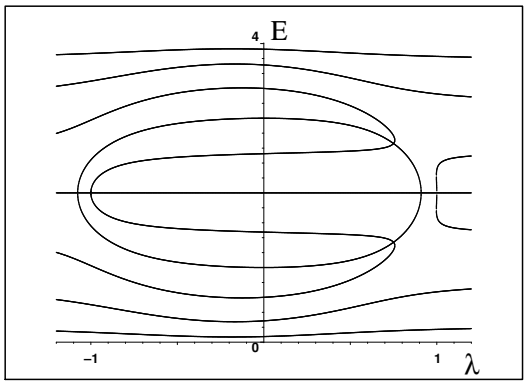


Figure 2: The $\lambda-$dependence of the eigenvalues of Hamiltonian (11) in which we selected the constantly shifted value of $\mu = \mu(\lambda) := \lambda + 0.25$.

One can easily check, in Fig. 2, that the original picture lost its left-right symmetry and that the real bound-state-energy values only occur in a smaller, asymmetric interval of $\lambda \in (-1, b)$ where $b \approx 0.75$ for our particular illustrative choice of the constant shift $\Delta = \mu - \lambda$. The further inspection of the picture reveals many further and qualitatively interesting features of the "phase transition" during which the pairs of individual energy levels cross or merge and, subsequently, complexify. Temporarily, some of the complexified pairs may even return to the reality later – notice, in the picture, that there are as many as nine real level at λs which lie slightly below the critical $\lambda = 1$.

3.2 Pseudometrics

In a way paralleling the preceding section we shall now restrict attention to the intervals of $\lambda \in (a(\mu), b(\mu))$ and $\mu \in (c(\lambda), d(\lambda))$, i.e., to the two-dimensional physical domain $\mathcal{D}$ of "acceptable" parameters in the Hamiltonian. Inside this domain the whole spectrum remains, by definition, completely real and non-degenerate, i.e., potentially, physical, observable and compatible with the unitarity of the time evolution.

In this setting the obligatory construction of the suitable matrices of the metric may proceed along the same lines as above. In full parallel, we shall therefore return to the independent variability of the two couplings in the Hamiltonian and reopen the problem of the construction of the metric via Eq. (7). In the language of Ref. [1], the $N-$parametric ambiguity contained in the latter formula makes the related picture of physics flexible and adaptable to our potential choice of further relevant operators (i.e., in our case, of some N by N matrices) of observables.

Under our present restricted project, we shall again pay attention merely to the explicit construction of the "most interesting" N by N pseudometric $\mathcal{P}^{(J)}$ at $J = 6$ and $N = 2J + 1 = 13$. The method of construction will remain the same. During its application we displayed, first of all, the non-vanishing matrix elements of the sparse difference matrix $H^\dagger \mathcal{P}^{(6)} - \mathcal{P}^{(6)} H$ and made them equal to zero via the solution of the corresonding algebraic equations. At the end of this procedure which completely paralleled our preceding use of ansatz (9) as well the format of result (10) we obtained the matrix elements of our sample pseudometric $\mathcal{P}^{(6)}$ in the following, equally compact and comparably transparent form

$$r = \frac{(1+\mu)\,(1-\lambda)}{1+\lambda^2+2\,\mu^2}\,, \qquad s = \frac{1+\mu}{1+\lambda^2+2\,\mu^2}\,, \qquad v = \frac{1}{1+\lambda^2+2\,\mu^2}$$

$$t=\frac{1+\lambda^2}{1+\lambda^2+2\,\mu^2}\,,\qquad w=\frac{1+\lambda^2+\mu^2}{1+\lambda^2+2\,\mu^2}\,.$$

One should add here that due to the multiple symmetries of our Hamiltonian matrix as well as of the metric, the inversion of the metric (or pseudometric) may be obtained by the simple change of the sign of the pair of our coupling-constant quantities λ and μ. The inspection of the latter formulae also reveals that the numerators remain the same so that they might be all omitted or ignored as an inessential overall multiplication factor.

4 Three-parametric Hamiltonians

For a proper, non-degenerate tractability of the next family of some three-parametric Hamiltonians we need to deal with the dimensions $N \geq 13$ at least. The full matrices will not fit in the printed page anymore. Fortunately, their numerous symmetries will still allow us to display the relevant information about their matrix elements. In particular, it proves sufficient to display just the upper part of the Hamiltonian matrix in full detail,

$$H^{(N)}=\left[\begin{array}{ccccccc}
2 & -1-\lambda & 0 & \ldots & & \ldots & 0 \\
-1+\lambda & 2 & -1+\mu & 0 & \ldots & & \vdots \\
0 & -1-\mu & 2 & -1-\nu & 0 & \ldots & \\
\vdots & 0 & -1+\nu & 2 & -1 & \ddots & \vdots \\
 & & \ddots & -1 & 2 & \ddots & 0 \\
\vdots & & & \ddots & \ddots & \ddots & -1+\lambda \\
0 & \ldots & \ldots & 0 & 0 & -1-\lambda & 2
\end{array}\right]\,.$$

Similarly, the symmetries of the most interesting $N = 13$ pseudometric component $\mathcal{P}^{(7)}$ of the $N = 13$ metric (9) enables us to search for its matrix elements via the thirteen-dimensional matrix ansatz

$$\mathcal{P}^{(7)} = \begin{bmatrix}
0 & \dots & & & \dots & 0 & r & 0 & \dots & & & \dots & 0 \\
\vdots & & & \dots & 0 & s & 0 & s & 0 & \dots & & & \vdots \\
 & & \dots & 0 & p & 0 & t & 0 & p & 0 & \dots & & \\
 & & \iddots & v & 0 & q & 0 & q & 0 & v & \ddots & & \\
 & & \iddots & 0 & w & 0 & m & 0 & w & 0 & \ddots & & \\
 & & \iddots & \iddots & 0 & u & 0 & u & 0 & \ddots & \ddots & & \\
 & & & \iddots & \iddots & 0 & 1 & 0 & \ddots & \ddots & & & \\
\vdots & & & & & & \vdots & & & & & & \vdots \\
0 & \dots & & \dots & 0 & 0 & r & 0 & 0 & \dots & & \dots & 0
\end{bmatrix} \tag{12}$$

It is worth adding that wherever we decide to choose $N > 13$, the triple dots may be read here as indicating, for all of the sharply larger dimensions, simply the repetition of the same (i.e., of the last) element until the symmetry of the matrix allows.

Strictly the same procedure as above leads again to the final and still amazingly compact solution

$$r = \frac{(1-\nu)\,(1+\mu)\,(1-\lambda)}{1+\lambda^2+2\,\mu^2+3\,\nu^2+\nu^2\lambda^2}\,, \qquad s = \frac{(1-\nu)\,(1+\mu)}{1+\lambda^2+2\,\mu^2+3\,\nu^2+\nu^2\lambda^2}$$

$$p = \frac{1-\nu}{1+\lambda^2+2\,\mu^2+3\,\nu^2+\nu^2\lambda^2}\,, \qquad v = \frac{1}{1+\lambda^2+2\,\mu^2+3\,\nu^2+\nu^2\lambda^2}$$

$$t = \frac{(1-\nu)\,(1+\lambda^2)}{1+\lambda^2+2\,\mu^2+3\,\nu^2+\nu^2\lambda^2}\,, \qquad q = \frac{1+\mu^2+\lambda^2}{1+\lambda^2+2\,\mu^2+3\,\nu^2+\nu^2\lambda^2}$$

$$w = \frac{1+\mu^2+\nu^2+\lambda^2}{1+\lambda^2+2\,\mu^2+3\,\nu^2+\nu^2\lambda^2}\,, \qquad m = 1-2\,\frac{\nu^2}{1+\lambda^2+2\,\mu^2+3\,\nu^2+\nu^2\lambda^2}$$

$$u = 1-\frac{\nu^2}{1+\lambda^2+2\,\mu^2+3\,\nu^2+\nu^2\lambda^2}$$

From this set of formulae we may extract the similar messages as above.

5 Discussion

In the sense of commentaries scattered over the preceding sections we now intend to complement the preceding Summary section by an outline of a few possible future mathematical and methodical as well as purely phenomenologically motivated extensions of the model.

In the corresponding list of the possible directions of a generalization of the present model, the one which looks most worth pursuing lies in the systematic search for the further exactly solvable finite-dimensional models which would admit not only the closed-form representation of the real spectrum of the energies but also the explicit construction of the metric operator. Even if one would be able to construct just some (i.e., not all) metrics (which is, after all, most common in the literature), the scarcity of the exactly solvable models in this field would certainly provide a ground for the publication of this type of the results.

By our recommendation one might particularly concentrate attention to the preservation of the localized support of the interactions near the corners of the tridiagonal Hamiltonian matrix. This idea was originally inspired by the discovery of the tractability of the differential-equation $N \to \infty$ models with point interactions at the boundaries [8]. At the finite dimensions, the same features of the dynamics have now been found to survive even in the models constructed at the not too large dimensions $N \ll \infty$. We believe, therefore, that the latter choice of the specific dynamics will gain further popularity as a ground of an optimal solvable-model-building strategy in the nearest future.

Certainly, there exist further interesting aspects of a systematic, model-based quantum mechanics of the elementary models which look non-Hermitian when solely considered in the most user-friendly, F-superscripted Hilbert

space $\mathcal{H}^{(F)}$. One of the most obvious apparent paradoxes may be seen in the mathematical non-uniqueness of the assignment of the metric Θ to a given Hamiltonian H. Fortunately, the answer has already been provided twenty years ago when the authors of Ref. [1] gave the complete answer. Briefly stated: the ambiguity $\Theta = \Theta(H)$ merely reflects the open possibility of incorporation of additional phenomenological information via an introduction of more observable quantities.

The best known illustrative example of such an added observable is the Bender's "charge" [4]. Now, whenever one chooses this charge or another observable as a phenomenological input, the possibility and feasibility of the construction of the complete family of the eligible metrics $\Theta = \Theta(H)$ in a closed, non-numerical form will always represent a significant advantage of the mathematical model. Plus, needless to add, the use of any analytic though still flexible form of the metric which appears in the mean values, i.e., in principle, which enters all of the measurable predictions would certainly enhance the appeal of the theory in applications.

Another apparent paradox concerns the "kinematical" multi-index parameter α which reflects the above-mentioned ambiguity and which numbers the alternative eligible metrics $\Theta(H) = \Theta_\alpha(H)$. It is obvious that for some values $\alpha_{critical}$ of these parameters the metric itself may become singular and unacceptable. An interesting potential reward of the further study of a particular quantum model characterized by an operator (or, in our case, matrix) doublet (H, Θ_α) might be seen in the possible quantitative specification of the connections between the critical values of $\alpha_{critical}$ as functions, say, of the (possibly, multi-index) dynamics-determining couplings λ in $H = H(\lambda)$.

In some sense, the closely related and/or complementary questions will also emerge in connection with any toy-model $H = H(\lambda)$ in which the complexification of the eigenenergies occurs at the so called Kato's exceptional

points $\lambda_{cricical}$ (at which the energies merge and subsequently complexify - for illustration see, e.g., the presence of the pair of exceptional points $\lambda_{cricical} = \pm 1$ in Figure 1). In particular, an explicit future construction of solvable models might be able to clarify the mutual connections between, firstly, the "dynamical" loss of the observability of the energies at $\lambda = \lambda_{cricical}$ and, secondly, the "kinematical" loss of the existence of the pre-selected S-superscripted Hilbert space at $\alpha = \alpha_{cricical}$ in connection, thirdly, with the necessary loss of the observability of some *other* dynamical observable at *the same* $\alpha = \alpha_{cricical}$ (the readers should consult, first of all, Ref. [1] in this context).

Last but not least, another natural future continuation of research which may be expected exceptionally promising might concentrate upon the scenario in which the eigenvalues of H remain real while the metric re-regularizes "insufficiently", becoming merely indefinite after the parameter α itself crosses, in an appropriate manner, the critical value of $\alpha_{cricical}$. In such a context, one might merely re-classify the resulting "wrong" or "indefinite" metric Θ_α as the Bender's "parity" $\mathcal{P}$ and search for his "charge" $\mathcal{C}$ in the "new metric" $\Theta_{changed} = \mathcal{PC}$ (cf. [4] for the complete recipe).

6 Summary

On the background of comparison with the older results of Paper 1, one of the most surprising features of their present generalization may certainly be seen in the friendly nature of the more-parametric formulae. A completion and further extension of such constructions along the lines indicated in preceding section seems to be a project with good chances for a success in the future, indeed.

Our present first results in this direction may be briefly summarized as

follows. Firstly, we revealed an emergent pattern of having, up to an overall factor, the purely polynomial matrix elements of the "pseudometric" components $\mathcal{P}$ of the metrics. Our sample calculations found such a hypothesis reconfirmed.

Secondly, we may feel impressed by the emergence of the pattern of most natural and obvious further generalizations of the Hamiltonians in which one introduces new and new parameters at an increasing distance from the boundaries of the lattice. It is certainly encouraging that such a recipe leaves the construction non-numerical and that it seems to offer unexpectedly compact and transparent benchmark-type results.

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